ACOUSTIC AND HYDRODYNAMIC PROCESSES IN A TRANSLATING NONINERTIAL REFERENCE FRAME

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Historically both linear and nonlinear acoustics has been developed mainly on the basis of the hydrodynamic equations in an inertial reference frame. The rare exceptions are certain problems in atmospheric hydrodynamics involving the generation and propagation of infrasound on a sphere rotating with a constant angular velocity (see [1], for example) and a limited number of examples in the theory of inertial hydropulsators [2]. Movable objects of finite size, which can move with acceleration (including time-dependent acceleration) are common in technology and hence it is necessary to consider acoustic and hydrodynamic phenomena in coordinate systems rigidly fixed to these objects.

1. We chose as a starting point the closed system of hydrodynamic equations for an Euler fluid (without sources) in a reference frame moving translationally with an acceleration $\mathbf{a} = \mathbf{a}(t)$ relative to an inertial reference frame:

$$\rho[\dot{\mathbf{v}} + (\mathbf{v}\nabla)\mathbf{v}] = -\nabla P - \rho\mathbf{a}; \tag{1.1}$$

$$\dot{\rho} + \operatorname{div}\rho\mathbf{v} = 0; \tag{1.2}$$

$$\dot{\sigma} + (\mathbf{v}\nabla)\sigma = 0; \tag{1.3}$$

$$P = P(\rho, \sigma); \tag{1.4}$$

$$T d\sigma = dw - dP/\rho. \tag{1.5}$$

Here $\mathbf{v}$ is the hydrodynamic velocity; ρ is the density of the medium; P is the hydrodynamic pressure; T is the absolute temperature; σ is the entropy per unit mass; w is the specific enthalpy, which can be expressed through the internal energy per unit mass $\mathscr{E}(\rho, \sigma)$ as $w = \mathscr{E} + P/\rho$. The system of equations (1.1) through (1.5) must be supplemented by the appropriate initial and boundary conditions.

It is to be noted that the equations of equilibrium thermodynamics strictly apply only to objects moving as a rigid whole in a straight line with constant velocity or rotating uniformly with respect to an inertial reference frame [3]. However if we assume an equation of state of the type (1.4) can be used in the noninertial frame, then the principles of equilibrium thermodynamics can be used without any further restrictions [4].

A qualitative analysis of the basic equations (1.1) through (1.5) shows that in general the medium is stratified in a noninertial reference frame. This occurs because of the presence of the term $-\rho\mathbf{a}$ on the right-hand side of (1.1), which characterizes the effect of the inertial force. However this stratification of the medium will only be stable when a is not a function of time. In this case, for a known functional dependence $w_S = w_S(P_S, \sigma_S)$ the basic system of equations (1.1) through (1.5) can be used to find the stratification law for the density (pressure) and entropy (see [5, 6], for example). Here the subscript s denotes the case $\dot{\mathbf{v}}_s = 0$ for $\mathbf{a} = \text{const}$. The problem will be significantly more complicated when $\mathbf{a} = \mathbf{a}(t)$. We assume that the acceleration of the noninertial reference frame can be written in the form

$$\mathbf{a}(t) = \langle\mathbf{a}\rangle + \mathbf{a}', \tag{1.6}$$

where $\mathbf{a}_0 = \langle\mathbf{a}\rangle$ is slowly varying in time and $\mathbf{a}'$ is rapidly varying. Here and below the symbol <...> denotes a time average of the form

$$\langle a(t)\rangle = \frac{1}{t_0}\int_t^{t+t_0} a(t')\,dt', \tag{1.7}$$

where the interval of time used in the averaging t_0 is large in comparison with the characteristic time of the rapid process τ_1 and is small in comparison with the characteristic

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time τ_2 of the slow process. A similar decomposition has been used in the past to study the hydrodynamics of a plasma in a strong high-frequency field. If the "high-frequency" component of the acceleration is absent ($a' = 0$) then one can speak of a stratified structure "floating" in time; the results of [5, 6] can be applied formally, where g (the gravitational acceleration) is replaced by a_0 and all of the thermodynamic variables will depend functionally on τ_2, in addition to the usual variables.

In the case when $\mathbf{a}_0$ = const, or $1/\tau_2 \gg N$ (here and below we assume that $\mathbf{a} = a\mathbf{e}_x$), where $N = \left\{-\langle a\rangle\left[\frac{1}{\langle\rho\rangle}\frac{\partial\langle\rho\rangle}{\partial x} + \frac{\langle a\rangle}{\langle c^2\rangle}\right]\right\}^{1/2}$ is a frequency of the Väisälä type [1], $c^2 = \partial P/\partial\rho$ is the square of the speed of sound, $\mathbf{e}_x$ is a unit vector along the x axis, and the average is understood in the sense of (1.7), a wave component is developed on the background equilibrium (or quasi-equilibrium) thermodynamic state when $a' \neq 0$, and the dispersion relation contains (in the absence of average fluxes) two types of waves (acoustic and internal). An analogous situation exists in atmospheric hydrodynamics [1].

2. The energetics of the hydrodynamic processes in a noninertial reference frame can be studied using the law of conservation of energy in the form of the Umov–Poynting theorem:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + \mathscr{E} + xa\right)\right] + \operatorname{div}\left[\rho\mathbf{v}\left(\frac{v^2}{2} + w + xa\right)\right] = \rho x\dot{a}(t) \tag{2.1}$$

[($E = \rho(v^2/2 + \mathscr{E} + xa)$ is the energy density in a noninertial reference frame [7], $\dot{a} = \partial a/\partial t$)]. Equation (2.1) can easily be verified by the standard method of direct differentiation [8]. The expression $\rho x\dot{a}(t)$ on the right-hand side of (2.1) characterizes the power of the inertial force per unit volume.

We consider only the acoustic wave field and neglect effects associated with the presence of internal waves. Then we can transform from a two-parameter equation of state $P = P(\rho, \sigma)$ to a one-parameter equation of the form $P = P(\rho)$. This approach will be valid if the frequency of the acoustic wave is larger than the frequency N (this includes the case $\langle a\rangle = 0$, which is important in practice) or if the characteristic linear dimension ℓ of the volume under consideration is much smaller than the scale of the inhomogeneities of the medium $L \sim \langle c^2\rangle/\langle a\rangle$. For example in air $\langle c^2\rangle \sim 9\cdot 10^4$ m^2/sec^2. Taking $\langle a\rangle \sim$ 0.3g-3g, we have L ~ 3-30 km and the above condition is obviously satisfied for objects of practical interest.

We consider conservation of energy of the acoustic wave field in the linear approximation. To do this the variables appearing in (1.1) through (1.5) and (2.1) are written in the form $\mathbf{v} = \langle\mathbf{v}\rangle + \mathbf{v}'$, $\rho = \langle\rho\rangle + \rho'$, $P = \langle P\rangle + P'$, $\mathbf{a} = \langle\mathbf{a}\rangle + \mathbf{a}'$, which separates the "fast" and "slow" parameters. Here we will consider the special (but important in practice) case when $\langle\mathbf{v}\rangle = 0$, $\langle\mathbf{a}\rangle = 0$, $\langle\rho\rangle = \rho_0 = \text{const}$, $\langle P\rangle = P_0 = \text{const}$.

We expand the left- and right-hand sides of (2.1) in the small parameters ρ', $\mathbf{v}'$, P', to terms of the second order, obtaining

$$\frac{\partial}{\partial t}\left\{[\rho_0(\mathscr{E}_0 + xa')] + [(w_0 + xa')\rho'] + \left[\frac{\rho_0 v'^2}{2} + \frac{c_0^2}{2\rho_0}\rho'^2\right]\right\} + \\ + \{\operatorname{div}[\rho_0\mathbf{v}'(w_0 + xa')]\} + \{\operatorname{div}[(P'\mathbf{v}' + \rho'\mathbf{v}'(w_0 + xa'))]\} = [\rho_0 x\dot{a}'] + [\rho' x\dot{a}'] \tag{2.2}$$

(c_0^2 is the mean square "background" speed of sound). In simplifying (2.2) it is necessary to use the identity $\frac{\partial}{\partial t}[\rho_0(\mathscr{E}_0 + xa')] \equiv \rho_0 x\dot{a}'$. The derivative of the linear part of the energy density of the medium

$$\frac{\partial}{\partial t}[\rho'(w_0 + xa')], \tag{2.3}$$

is not identically cancelled by the term $[\rho' x\dot{a}']$ on the right-hand side of (2.2) because higher-order terms of the form $[-(w_0 + xa')\operatorname{div}\rho'\mathbf{v}']$ appear when the differentiation is carried out. Hence the "source" term $\rho' x\dot{a}'$ must be attributed to both the linear and quadratic (in our approximation) parts of the expansion of (2.2). Simplifying (2.2) with the above discussion taken into account, and integrating over the entire noninertial space, we can write

$$\frac{\partial}{\partial t}\int d^3\mathbf{x}\left(\frac{\rho_0 v'^2}{2} + \frac{c_0^2\rho'^2}{2}\right) + \oint dS\cdot P'v'_n = 0, \tag{2.4}$$

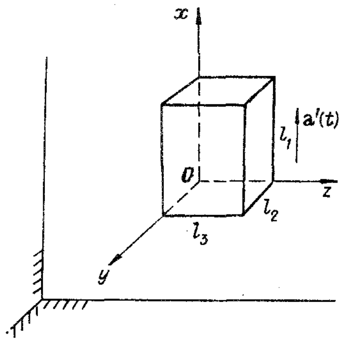

Fig. 1

where v_n' is the component of the acoustic velocity along the outward normal; S is the surface surrounding the volume of integration. In the derivation of (2.4) the Ostrogradskiĭ–Gauss theorem was used and we made the realistic assumption that the total mass and momentum fluxes integrate to zero over all space. This assumption is essentially equivalent to an averaging of the terms with the dimensions of energy over the spatial coordinates and an averaging of the momentum flux with respect to time (see [9]). From the conservation of energy of the wave field (2.4) it follows that the "source" term $\rho x\dot{a}(t)$ in (2.1) cannot lead to excitation of sound in an infinite space. This is because the inertial force causes a coherent motion of the particles of the medium, i.e., motion of the medium as a whole. We note that the application of (2.4) to a noninertial reference frame has apparently not been discussed before.

3. We now consider boundary-value problems. We linearize the system (1.1) through (1.5) in the acoustic approximation, assuming that $P = P(\rho)$ and using (1.6). We then obtain the wave equation for the pressure perturbation $P' = P - \langle P\rangle$:

$$\Delta P' - \frac{1}{c_0^2}\frac{\partial^2 P'}{\partial t^2} = 0. \tag{3.1}$$

In order to be definite, we specify the geometry of the problem. Let a parallelepiped with sides ℓ_1, ℓ_2, ℓ_3 and perfectly rigid walls be accelerated in the x direction, along the side of length ℓ_1 (Fig. 1). Here and below for simplicity we will assume that $\langle a\rangle = 0$, $a' = b_0 \sin\omega_0 t$ (b_0 is the amplitude of the variable part of the acceleration of the moving volume and ω_0 is the oscillation frequency). The boundary conditions for P' in (3.1) on the sides of the parallelepiped are unknown a priori. To overcome this difficulty, we transform to a wave equation for the acoustic velocity potential.

$$\Delta\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{x b_0\omega_0}{c_0^2}\cos\omega_0 t. \tag{3.2}$$

Equations (3.1) and (3.2) will be solved assuming a narrow volume, i.e., ℓ_2, $\ell_3 \ll \lambda$ (λ is the wavelength of the sound wave). In this approximation $P' = P'(x, t)$, i.e., the pressure perturbation does not depend on y and z. This approximation is well-known from the theory of forced vibrations in narrow pipes [10]. Also the frequency of the forced vibrations is equal to the frequency ω_0 of the driving force.

The solution of (3.2) with the boundary conditions $v_x'|_{x=0,l_1} = \frac{\partial\varphi}{\partial x}\Big|_{x=0,l_1} = 0$ has the form $\varphi = \left[x + \frac{1}{k}\frac{\sin k\left(\frac{l_1}{2} - x\right)}{\cos\frac{kl_1}{2}}\right]\frac{b_0}{\omega_0}\cos\omega_0 t$, where $k = 2\pi/\lambda = \omega_0/c_0$ is the wave number. Using the relation $P' = -\rho_0 a' x - \rho_0\partial\varphi/\partial t$ we obtain the solution

$$P'(x, t) = \frac{b_0\rho_0}{k}\frac{\sin k\left(\frac{l_1}{2} - x\right)}{\cos\frac{kl_1}{2}}\sin\omega_0 t, \tag{3.3}$$

which satisfies the wave equation (3.1) with the boundary conditions $P'|_{x=0} = \frac{b_0\rho_0}{k}\tan\frac{kl_1}{2}\sin\omega_0 t$, $P'|_{x=l_1} = -\frac{b_0\rho_0}{k}\tan\frac{kl_1}{2}\sin\omega_0 t$. The physical meaning of these boundary conditions is obvious. In a translationally moving noninertial reference frame an inertial force directed opposite to the acceleration a' acts on a particle of the medium. Hence the pressure is a minimum on the end $x = \ell_1$ in Fig. 1 and is a maximum on the opposite end ($x = 0$ in Fig. 1).

We consider the long-wavelength (low-frequency) approximation, when the parameter $k\ell_1$ is small. Expanding (3.3) in a series in $k\ell_1$ and keeping terms up to the third order, we obtain

$$P' \simeq \frac{b_0\rho_0}{k}\left[k\left(\frac{l_1}{2}-x\right)-\frac{k^3\left(\frac{l_1}{2}-x\right)^3}{6}+\frac{k^3 l_1^2\left(\frac{l_1}{2}-x\right)}{8}\right]\sin\omega_0 t. \tag{3.4}$$

In this approximation the pressure perturbation P' can be decomposed into two parts: an acoustic part π and a pulsating part p: $P' \simeq \pi + p$, $\pi \simeq \frac{b_0\rho_0 k^2}{2}\left(\frac{l_1}{2}-x\right)\left[\frac{l_1^2}{4}-\frac{\left(\frac{l_1}{2}-x\right)^2}{3}\right]\sin\omega_0 t$, $p \simeq b_0\rho_0\left(\frac{l_1}{2}-x\right)\sin\omega_0 t$. The pulsating part of the pressure perturbation represents an in-phase variation of pressure at every point in the volume (pseudosound). In the limit of an incompressible medium ($c_0 \to \infty$) we have $P' \to p$.

We estimate the components of the pressure in (3.4):

$$|p/\pi| \sim 10\,(kl_1)^{-2}. \tag{3.5}$$

Let the characteristic linear dimension of the moving volume be $\ell_1 \sim 1$ m. Then the pulsating part of the pressure will be dominant at frequencies $\omega \ll 300$ sec^{-1} (for air), $\omega \ll 1500$ sec^{-1} (for water), i.e., in the infrasonic region. Using (3.5), we estimate the pressure inside the body of a light automobile for the most typical frequency interval of $\omega \sim 36$-130 sec^{-1} (see [11]): $|p/\pi| \sim 10^3$-10^2. For example, for $\ell_1 = 1$ m, $\omega = 100$ sec^{-1} we obtain $|p/\pi| \simeq 131$ at $x = 0$.

We return now to the case when $a'(t)$ is an arbitrary function of time and $\langle a\rangle = 0$ and we limit ourselves to the infrasonic range of frequencies. For the geometry of the problem considered here (Fig. 1) we can write

$$P' \simeq \pi + p \simeq \pi - a'\rho_0(x - l_1/2). \tag{3.6}$$

We compare the components of the pressure in (3.6). Putting $\pi \sim 10^{-4}$-10^2 Pa, $(x - \ell_1/2) \sim 1$ m, $\rho_0 \sim 1.3$ kg/m^3 (for air), $\rho_0 \sim 10^3$ kg/m^3 (for water), we find the range of accelerations a' for which $|p| \gg |\pi|$. We obtain $a' \sim 10^{-4}$-10^2 m/sec^2 (for air), $a' \sim 10^{-7}$-10^{-1} m/sec^2 (for water). Hence for typical values of the variable part of the acceleration of the moving object the contribution of the pulsating part of the pressure can be dominant. In the case of water we have $|p/\pi| \sim 10$-10^7 even for $a' = 1$ m/sec^2.

In the approximation considered here (3.1) can be transformed to $\Delta\pi - \frac{1}{c_0^2}\frac{\partial^2\pi}{\partial t^2} = \frac{\rho_0 x}{c_0^2}\frac{\partial^2 a'}{\partial t^2}$, which is a wave equation for the acoustic pressure in the infrasonic frequency region.

In the more general case the pressure perturbation P' can be decomposed into the components p and π since in the limit $P' \to p$ (corresponding to an incompressible medium) the wave equation (3.1) transforms into Laplace's equation $\Delta p = 0$.

Finally we note that our analysis can be useful in the design of experiments and in analyzing acoustic measurements carried out in moving objects.

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DISCRETE CHARACTER OF THE FORMATION OF VORTICES IN A DEVELOPING CIRCULATORY FLOW

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The problem of a developing circulatory flow past an airfoil impulsively brought from rest to a constant velocity in an inviscid, incompressible fluid was quantitatively solved for the first time (in the linear approximation) in the work of Wagner [1] more than 60 years ago. This work was based on the Prandtl's assumption of the continuous vortex shedding from the sharp trailing edge of an airfoil. The more general case of the unsteady problem of flow past a moving airfoil associated with the occurrence of flutter attracted, more than 50 years ago, the attention of Soviet scientists M. V. Keldysh, M. A. Lavrent'ev, A. I. Nekrasov, and L. I. Sedov, who developed the existing standard linear theory of the unsteady motion of an airfoil. The difficulties in the nonlinear problem, and their early concepts, are still valid today for the exact description of the flow and were elaborated by Sedov [2] 50 years ago.

Because of the difficulties in the exact description of the flow, we can replace the exact description of the vortex shedding from the trailing edge by a model. The problem of the developing circulatory flow past an airfoil impulsively brought from rest to a constant velocity was solved in [3, 4]. The flow past a flat plate with an angle of incidence $\alpha = 90°$ was examined by means of a dipole model under the assumption that the flow does not separate near the leading edge. It is emphasized that the dipole field represents not only the limiting case of a source–sink system but also the limiting case (in the direction perpendicular to the dipole axis) of a system of two vortices with velocity circulation of opposite signs. Therefore, the dipole model applied to describe flow containing domains with closed streamlines can be viewed as a degenerate classic Föppl's model with vortices of infinite circulation located on the surface of a body.

The above works show that, after approaching some critical instant of time t_*, the streamline which passes through the trailing edge no longer encloses the domain where trajectories of fluid elements form closed lines, and it was assumed that for $t > t_*$ this domain separates from the plate. Also, it was assumed that the new domain with closed trajectories of fluid elements was formed on the trailing edge after elapse of a period of time. The new domain grows until it reaches again a critical size at $t = t_{**}$, and so on.

Thus, the process of development of circulatory flow past an airfoil in an inviscid fluid characterized by vortex shedding from the trailing edge is not continuous but consists of subsequent formations and separations of domains with closed trajectories of fluid elements formed by discrete elements of the vortex sheet.

The supplementary information on the dipole model for $t = t_*$ presented below helps to estimate the circulation of the first vortex separated from the trailing edge of the plate as well as to examine the pattern of the flow for $t > t_*$ after bifurcation of a dipole.

The instantaneous patterns of flow for $t < t_*$ in the system of coordinates associated with the plate are shown in Fig. 1 for two descriptions of the flow. For an exact description the domain with closed trajectories of fluid elements is formed by the curled vortex sheet separating from the trailing edge of the plate (Fig. 1a). For the model description the analogous domain is formed by a dipole located at point D on the trailing edge of the

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